UDC 539.3

ASYMPTOTICS OF THE PRESSURE FIELD IN A SPECIFIC DISCONTINUITY FOR DIFFRACTION OF A PLANE ACOUSTIC WAVE ON AN ELASTIC CYLINDRICAL SHELL

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We consider a two-dimensional unsteady problem of diffraction of a plane acoustic wave on a closed, thin, elastic, circular cylindrical shell. We use the method of double integral transformations; Laplace transformation with respect to time, and Fourier transformation with respect to an angle, and obtain the inverse transforms in approximate form using the method of steepest descent and of determining residues at the poles. We derive the asymptotic formulas for the pressure fields behind the fronts of the emitted, reflected and diffracted waves, and analyze the influence of the shell elasticity on all the above waves. The state of the problem is elucidated in the monographs and articles [1 — 24] and the mathematical apparatus used here is that developed by Friedlander [1, 2, 13].

1. Statement of the problem. A closed, thin, elastic, circular cylindrical shell is at rest in an unbounded, ideal compressible fluid, and an acoustic wave

$$p_1(\xi, t) = p_0 g(t_\xi) H(t_\xi), \quad t_\xi = t + \xi \tag{1.1}$$

impinges on this shell along the normal to its longitudinal axis.

Let us consider the following two-dimensional unsteady problem: to find the pressure field in the wave p_2 (p_2 is the sum of the emitted, reflected and diffracted waves) caused by the action of the wave p_1 and satisfying the following wave equation as well as the initial and boundary conditions:

$$\left(\nabla_0^2 - \frac{\partial^2}{\partial t^2}\right) p_2 = 0, \quad \nabla_0^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1.2}$$

$$r \geqslant 1, \quad -\pi \leqslant \theta \leqslant \pi, \quad t \geqslant 0$$

$$p_2 = \frac{\partial}{\partial t} p_2 = u_j = \frac{\partial}{\partial t} u_j = 0 \quad (j = 1, 2, 3), \quad t = 0$$

$$q_r = -(p_1 + p_2), \quad \frac{\partial^2}{\partial t^2} u_3 = -\frac{1}{\rho C^2}\frac{\partial}{\partial r}(p_1 + p_2), \quad r = 1, \quad p_2 \to 0, \quad r \to \infty$$

We assume that the solution is bounded within its domain of definition.

The time is counted from the initial instant when the incident wave comes in contact with the shell surface at the point with coordinates $r = 1$, $\theta = 0$.

The equations of motion of the shell correspond to the linear, Timoshenko-type theory

$$L_{ij} u_j = q\delta_{i3} \quad (L_{ij} = L_{ji}, \quad i, j = 1, 2, 3) \tag{1.3}$$

$$L_{11} = \left(\frac{\partial^2}{\partial \theta^2} - \lambda^2 \frac{\partial^2}{\partial t^2}\right) - \varkappa^2, \quad L_{12} = \varkappa^2, \quad L_{13} = (1 + \varkappa^2)\frac{\partial}{\partial \theta}$$

$$L_{22} = a^2\left(\frac{\partial^2}{\partial \theta^2} - \lambda^2 \frac{\partial^2}{\partial t^2}\right) - \varkappa^2, \quad L_{23} = -\varkappa^2 \frac{\partial}{\partial \theta}$$

$$L_{33} = -\left(\varkappa^2 \frac{\partial^2}{\partial\theta^2} - \lambda^2 \frac{\partial^2}{\partial t^2}\right) + 1$$

In the above formulas (1. 1)—(1. 3) we use the following notation:

$$\xi = \frac{X}{R_0}, \quad r = \frac{R}{R_0}, \quad t = \frac{CT}{R_0}, \quad u_1 = \frac{U_\theta}{R_0}, \quad u_3 = \frac{U_r}{R_0}$$

$$\varkappa^2 = \left(\frac{C_2}{C_1}\right)^2, \quad \lambda^2 = \left(\frac{C}{C_1}\right)^2, \quad \mu^2 = \left(\frac{C_2}{C}\right)^2, \quad q = \frac{q_r \delta_1}{\rho C^2}$$

$$\delta_1 = \frac{1}{2}\delta\lambda^2, \quad \delta = \frac{\rho}{\rho_1}\frac{R_0}{h}, \quad a^2 = \frac{1}{3}\left(\frac{h}{R_0}\right)^2,$$

$$C_1^2 = \frac{E}{\rho_1(1-\nu^2)}, \quad C_2^2 = \frac{Ek_T}{2\rho_1(1+\nu)}$$

Here R and θ denote the radial and angular coordinates, T is time, X is the coordinate pointing in the direction opposite to the direction of propagation of the incident wave; ρ and C are the fluid density and speed of sound in the fluid; E, ν and ρ_1 denote the modulus of elasticity, Poisson's ratio and the shell material density, respectively; R_0 and $2h$ are the radius and the thickness of the shell; U_θ and U_r denote the tangential and radial components of the shell displacement vector, u_2 is the angle between the normal and the median shell surface, C_1 and C_2 are the velocities of propagation of the elastic wavefronts in the linear theory of thin Timoshenko-type shells, k_T is the numerical shear coefficient, p_0 is a constant with the dimension of pressure, g is an arbitrary bounded function defining the law of pressure variation in the incident wave; H is the unit Heaviside function and δ_{ij} is the Kronecker delta.

2. Formal solution. We obtain a solution of the problem using the method of integral transformations. We perform the integral Laplace transformation over the time t, and the integral Fourier transformation over the angle θ. Let us write the formulas for the forward and inverse transformations

$$f^L(r, \theta, s) = \int_0^\infty f(r, \theta, t)e^{-st}\, dt, \quad f^{LF}(r, \omega, s) = \int_{-\infty}^{\infty} f^L(r, \theta, s)e^{-i\omega\theta}\, d\theta$$

$$f^L(r, \theta, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^{LF}(r, \omega, s)e^{i\omega\theta}\, d\omega, \quad f(r, \theta, t) =$$

$$\frac{1}{2\pi i}\int_{a_0-i\infty}^{a_0+i\infty} f^L(r, \theta, s)e^{st}\, ds, \quad a_0 > 0$$

Here s denotes the Laplace transformation parameter and ω is the Fourier transformation parameter.

We carry out the Fourier transformation over the angle θ under the assumption that the function $f(r, \theta, t)$ is defined not only in the physical domain $r \geqslant 1$, $-\pi \leqslant \theta \leqslant \pi$, $t \geqslant 0$, but also on the Riemannian surface the sheets of which are defined by the formula

$$(2k-1)\pi \leqslant \theta \leqslant (2k+1)\pi \quad k = \ldots, -1, 0, 1, \ldots$$

Since we know in advance that the function $f(r, \theta, t)$ undergoes discontinuities on certain space-time characteristics, we shall regard $f(r, \theta, t)$ as a generalized function (distribution in the Schwartz sense). Using the displacement theorem, the relation $\xi = 1 - r\cos\theta$ and the integral representation

$$I_\omega(sr) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(sr\cos\theta - i\omega\theta)\,d\theta$$

we obtain the following LF transformation of the incident wave:

$$p_1^{LF}(r, \omega, s) = -A_0(s)\,I_\omega(sr), \quad A_0(s) = 2\pi p_0\, g^L(s)\, e^{-s} \tag{2.1}$$

For the zero initial conditions the LF transformation of the wave equation (1.2) assumes the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - s^2\left(1 + \frac{\omega^2}{s^2r^2}\right)\right]p_2^{LF} = 0 \tag{2.2}$$

Let us write the solution of (2.2) bounded for $r \to \infty$, in the form

$$p_2^{LF}(r, \omega, s) = A_2(s)\,K_\omega(sr) \tag{2.3}$$

where $K_\omega(sr)$ and $I_\omega(sr)$ are the modified Bessel functions and $A_2(s)$ is an undefined coefficient.

The LF transformations of the equations of motion of the shell (1.3) and of the conditions of contact at $r = 1$, together constitute the following system of algebraic equations:

$$(2.4)$$

$$a_{ij}^{LF}u_j^{LF} = -\frac{\delta_1}{\rho C^2}(p_1^{LF} + p_2^{LF})\big|_{r=1}\delta_{i3}, \quad a_{ij}^{LF} = a_{ji}^{LF} \quad (i, j = 1, 2, 3)$$

$$u_3^{LF} = -\frac{1}{s^2\rho C^2}\frac{\partial}{\partial r}(p_1^{LF} + p_2^{LF})\big|_{r=1}$$

$$a_{11}^{LF} = -(\omega^2 + s^2\lambda^2 + \varkappa^2), \quad a_{12}^{LF} = \varkappa^2, \quad a_{13}^{LF} = i\omega(1 + \varkappa^2)$$

$$a_{22}^{LF} = -[a^2(\omega^2 + s^2\lambda^2) + \varkappa^2], \; a_{23}^{LF} = -i\omega\varkappa^2, \; a_{33}^{LF} = \omega^2\varkappa^2 + s^2\lambda^2 + 1$$

We solve the above system by substituting (2.1) and (2.3) into (2.4), thus obtaining the following LF transformation for the pressure p_2:

$$p_2^{LF}(r, \omega, s) = A_0(s)\,K_\omega(sr)\,\frac{D_3 I_\omega'(s) - \delta_1 s D_2 I_\omega(s)}{D_3 K_\omega'(s) - \delta_1 s D_2 K_\omega(s)} \tag{2.5}$$

$$D_3 = \det\|a_{ij}^{LF}\| \quad (i, j = 1, 2, 3), \quad D_2 = \det\|a_{ij}^{LF}\| \quad (i, j = 1, 2)$$

Here and in the following a prime denotes the derivative with respect to the argument.

3. Olver expansions and the asymptotics of the LF transformations. Since the process of inverting the formal solution (2.5) is difficult when the parameters ω and s are arbitrary, we shall now construct the asymptotics of the formal solution under the assumption that the Laplace transformation parameter s is large, real and positive. We replace the modified Bessel functions and their first derivatives by the asymptotic Olver series [25 — 27]. Substituting these series into (2.5) and assuming the parameter s to be real and large, we obtain

$$p_2^{LF} \sim -A_0(s)\,(2\pi sr)^{-1/2}\varkappa_1\varkappa_2\,[1 - s^{-1}(\varkappa_3 + \varkappa_4) + O(s^{-2})] \times \exp(s\varkappa_5) \tag{3.1}$$

$$\varkappa_1 = \left(1 + \frac{\omega^2}{s^2r^2}\right)^{-1/4}, \quad \varkappa_2 = \frac{(s^2 + \omega^2)^{1/2}D_3 - \delta_1 s^2 D_2}{(s^2 + \omega^2)^{1/2}D_3 + \delta_1 s^2 D_2}$$

$$\varkappa_3 = \frac{1}{12}\left[9 - 7\frac{\omega^2}{s^2}\left(1+\frac{\omega^2}{s^2}\right)^{-1}\right]\left(1+\frac{\omega^2}{s^2}\right)^{-1/2}$$

$$\varkappa_4 = \frac{1}{24r}\left[3 - 5\frac{\omega^2}{s^2r^2}\left(1+\frac{\omega^2}{s^2r^2}\right)^{-1}\right]\left(1+\frac{\omega^2}{s^2r^2}\right)^{-1/2}$$

$$\varkappa_5 = -r\left(1+\frac{\omega^2}{s^2r^2}\right)^{1/2} + 2\left(1+\frac{\omega^2}{s^2}\right)^{1/2} + \frac{\omega}{s}\operatorname{arsh}\frac{\omega}{sr} - 2\frac{\omega}{s}\operatorname{arsh}\frac{\omega}{s}$$

Below we shall show that the above inverse LF transformation corresponds to the asymptotics of the reflected and emitted waves near the front. By the emitted waves we mean the waves generated in the fluid by the process of wave propagation taking place within the elastic shell.

4. Langer's representation and the asymptotics of the LF transformations of the diffracted waves with $s \to \infty$.

Consider the transition region in which

$$\omega \sim i\,[s + \alpha s^{1/3} + O(s^{-1/3})], \quad |\alpha| \sim 1 \tag{4.1}$$

and replace in it the modified Bessel functions and their first derivatives by the Langer's asymptotic formulas [2, 28]. Using these formulas and the expression for the Wronskian of the modified Bessel functions we can show that for large, real values of the parameter s the LF transformation (2.5) assumes the form

$$p_2^{LF}(r,\omega,s) \sim \frac{A_0(s)\,K_\omega(sr)\,I_\omega'(s)}{D_3K_\omega'(s) - \delta_1 sD_2K_\omega(s)}\,[1 + O(s^{-2/3})] \tag{4.2}$$

Below we shall show that the inverse of the LF transformation (4.2) corresponds to the asymptotics of the diffracted waves at the front. By diffracted waves we mean the waves generated in the fluid by the passage of the incident wave bending around the shell.

5. Inversion of the Fourier transformation.

5.1. Reflected and emitted waves. We write the inverse Fourier transformation of the LF transformation (3.1) in the form

$$p_2{}^L(r,\theta,s) \sim A_3(s)\,I(s), \quad A_3(s) = p_0 g^L(s)\,e^{-s}(2\pi sr)^{-1/2} \tag{5.1}$$

$$I(s) = \int_{-\infty}^{\infty} K(\omega)\exp[sk(\omega)]\,d\omega$$

$$K(\omega) = \varkappa_1\varkappa_2\,[1 - s^{-1}(\varkappa_3+\varkappa_4) + O(s^{-2})], \quad k(\omega) = \varkappa_5 + i\omega\theta s^{-1}$$

The integrand function has simple poles determined by the roots of the denominator in the expression for $\varkappa_2$ in (3.1). For large, real and positive values of the parameter s the coordinates of the poles can be written in the form of asymptotic expansions in inverse powers

$$\omega_1 \sim \pm i\lambda s\,[1 + r_1 s^{-2} + t_1 s^{-3} + O(s^{-4})], \quad r_1 = \frac{1}{2\lambda^2} \tag{5.2}$$

$$t_1 = -\frac{\delta_1}{2\lambda^4\sqrt{1-\lambda^2}}$$

$$\omega_2 \sim \pm i\lambda s\,[1 + r_2 s^{-2} + t_2 s^{-3} + O(s^{-4})], \quad r_2 = r_1\frac{\varkappa^2}{a^2(1-\varkappa^2)}$$

$$t_2 = t_1\frac{\varkappa^4}{a^2(1-\varkappa^2)^2}$$

$$\omega_3 \sim \pm \frac{i\lambda s}{\varkappa}\,[1 + r_3 s^{-1} + O(s^{-2})], \quad r_3 = r_1\,\frac{\delta_1 \varkappa}{\sqrt{\varkappa^2 - \lambda^2}}$$

The poles define the relative velocities of propagation of the elastic wavefronts in the shell : λ^{-1} for the membrane (ω_1) and flexural (ω_2) waves, $\varkappa\lambda^{-1}$ for the shear (ω_3) wave, and the critical angles of appearance of the emitted waves

$$\theta_{1*} = \theta_{2*} = 2\arcsin\lambda - \arcsin\frac{\lambda}{r}, \quad \theta_{3*} = 2\arcsin\frac{\lambda}{\varkappa} - \arcsin\frac{\lambda}{\varkappa r}$$

We shall distinguish three separate zones, depending on the angle of observation θ

$$0 < \theta \leqslant \theta_{1*}, \quad 2)\ \theta_{1*} < \theta \leqslant \theta_{3*}, \quad 3)\ \theta_{3*} < \theta$$

In the illuminated region $0 < |\theta| < \pi/2$ the L-transformation of the reflected wave determined in the approximate form from (5.1) by the method of steepest descent, it exists in all three zones. When the angle of observation θ increases and exceeds θ_{1*}, then the L-transformation of the reflected wave must be supplemented by the L-transformations corresponding to the emitted waves generated by the membrane and flexural waves propagating through the shell. When the angle of observation exceeds θ_{3*}, we must also include the L-transformation of the emitted wave generated by the shear wave propagating through the shell. The L-transformations of all emitted waves can be found by evaluating the residues at the poles of the integrand function (5.1).

The method of steepest descent enables us to write the integral in (5.1) in the form of a series in powers of the large parameter s

$$I(s) \sim (2\pi)^{1/2}[sk^{\mathrm{II}}(\omega)]^{-1/2}K(\omega)\exp[sk(\omega)]\left[1 + s^{-1}\sum_{j=1}^{4}\varphi_j(\omega) + O(s^{-2})\right]\Bigg|_{\omega=\omega_*} \quad (5.3)$$

$$\varphi_1(\omega) = \frac{1}{2}\,\frac{k^{\mathrm{III}}(\omega)\,K^{\mathrm{I}}(\omega)}{k^{\mathrm{II}}(\omega)\,K(\omega)}, \quad \varphi_2(\omega) = \frac{1}{8}\,\frac{k^{\mathrm{IV}}(\omega)}{k^{\mathrm{II}}(\omega)}$$

$$\varphi_3(\omega) = \frac{5}{24}\,\frac{[k^{\mathrm{III}}(\omega)]^2}{[k^{\mathrm{II}}(\omega)]^3}, \quad \varphi_4(\omega) = -\frac{1}{2}\,\frac{K^{\mathrm{II}}(\omega)}{k^{\mathrm{II}}(\omega)\,K(\omega)}$$

The coordinate of the point $\omega = \omega_*$ of steepest descent is given by the equation $k'(\omega) = 0$, and has the form

$$\omega_* = is\,\sin\beta, \quad \omega_* = is\sin\gamma, \quad \theta + \gamma - 2\beta = 0 \quad (5.4)$$

Figure 1 gives the geometric interpretation of the angles θ, β and γ where β is the angle of incidence and θ is the angle of observation.

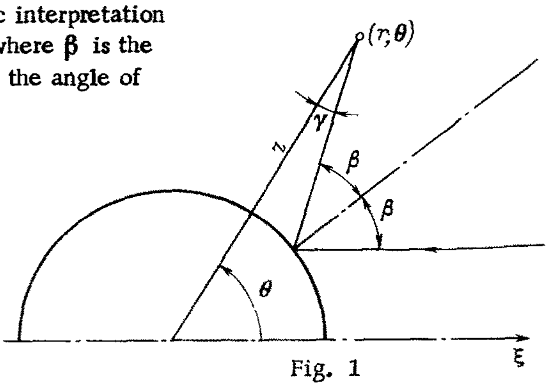

Fig. 1

Substituting the formulas (5.3) and (5.4) into (5.1) and performing simple manipulations, we obtain

$$p_2{}^L(r,\ \theta,\ s) \sim p_* g^L(s)\, e^{-sa_*}\left[1 - \frac{m}{s + \frac{1}{2}m} + ns^{-1} + O(s^{-2})\right] \qquad (5.5)$$

$$p_* = p_0\left(2r\frac{\cos\gamma}{\cos\beta} - 1\right)^{1/2}, \quad a_* = 1 - 2\cos\beta + r\cos\gamma$$

$$m = \delta\,[(1 - \mu^2\sin^2\beta)\cos\beta]^{-1}, \quad n = \sum_{j=1}^{5}\varphi_j(\omega_*)$$

$$\varphi_j(\omega_*) = \varphi_0(\omega_*)\,\psi_j(\omega_*) \quad (j = 1, 2, 3, 4)$$

$$\varphi_0(\omega_*) = \left[2\left(\frac{2}{\cos\beta} - \frac{1}{r\cos\gamma}\right)\right]^{-2}, \quad \psi_1(\omega) = \frac{\sin\gamma}{r\cos^2\gamma}\left(2\frac{\sin\beta}{\cos^3\beta} - \frac{\sin\gamma}{r^2\cos^3\gamma}\right)$$

$$\psi_2(\omega_*) = \frac{1}{2}\left(2\,\frac{1 + 3\,\mathrm{tg}^2\beta}{\cos^3\beta} - \frac{1 + 3\,\mathrm{tg}^2\gamma}{r^3\cos^3\gamma}\right)$$

$$\psi_3(\omega_*) = -\frac{5}{6}\left(2\frac{\sin\beta}{\cos^3\beta} - \frac{\sin\gamma}{r^2\cos^3\gamma}\right)^2\left(\frac{2}{\cos\beta} - \frac{1}{r\cos\gamma}\right)^{-1}$$

$$\psi_4(\omega_*) = -\frac{1 + \frac{5}{2}\mathrm{tg}^2\beta}{r^2\cos^2\beta}\left(\frac{2}{\cos\beta} - \frac{1}{r\cos\gamma}\right)$$

$$\varphi_5(\omega_*) = -\frac{1}{12\cos\beta}(9 + 7\,\mathrm{tg}^2\beta) - \frac{1}{24r\cos\gamma}(3 + 5\,\mathrm{tg}^2\gamma)$$

The Cauchy theorem on residues at the poles yields the L-transformations of the emitted waves. Substituting into (3.1) the coordinates of the poles ω_j $(j = 1, 2, 3)$ given by (5.2), we obtain

$$\mathrm{Res}\,[p_2{}^L(r, \theta, s);\ \omega_j] = A_0(s)\,(2\pi s)^{-1/2}\varkappa_1\beta_1\beta_2^{-1}\ [1 - s^{-1}(\varkappa_3 + \varkappa_4) + O(s^{-2})]\ e^{sk(\omega)}\,|_{\omega=\omega_j}$$

$$\beta_1 = (s^2 + \omega^2)^{1/2}D_3 - \delta_1 s^2 D_2, \quad \beta_2 = i\frac{\partial}{\partial\omega}[(s^2 + \omega^2)^{1/2}D_3 + \delta_1 s^2 D_2]$$

Neglecting terms of the order of a^2 and s^{-2} which are small compared with unity and performing simple manipulations, we obtain the following L-transformations of the emitted waves:

$$p_2{}^L(r, \theta, s) = \sum_{j=1}^{3} P_j g^L(s)\, e^{sa_j}\,[m_j s^{-(\sigma_j + 1/2)} + n_j s^{-(\sigma_j + 3/2)} + O(s^{-(\sigma_j + 5/2)})] \qquad (5.6)$$

$$P_1 = P_2 = (2\pi)^{1/2}(r^2 - \lambda^2)^{-1/4}p_0, \quad P_3 = (2\pi\varkappa)^{1/2}(\varkappa^2 r^2 - \lambda^2)^{-1/4}p_0$$

$$m_1 = -\frac{\delta_1}{\lambda^3\sqrt{1 - \lambda^2}}, \quad m_2 = -\frac{\varkappa^4\delta_1}{a^2\lambda^3(1 - \varkappa^2)^2\sqrt{1 - \lambda^2}}$$

$$m_3 = \frac{\delta_1}{\lambda\sqrt{\varkappa^2 - \lambda^2}}, \quad n_3 = -\frac{\varkappa^2}{2a^2\lambda^2(\varkappa^2 - 1)}$$

$$\sigma_1 = \sigma_2 = 2, \quad \sigma_3 = 0, \quad a_1 = -l_1 - \frac{1}{2}z_1\lambda^{-1}s^{-2}$$

$$a_2 = -l_2 - \frac{1}{2} \varkappa^2 z_2 \lambda^{-1} a^{-2} (1 - \varkappa^2)^{-1} s^{-2}$$

$$a_3 = -l_3 - \frac{1}{2} \delta_1 z_3 \lambda^{-1} (\varkappa^2 - \lambda^2)^{-1/2} s^{-1}$$

$$l_1 = l_2 = 1 + \lambda z_1 + \sqrt{r^2 - \lambda^2} - 2\sqrt{1 - \lambda^2}$$

$$l_3 = 1 + \frac{\lambda}{\varkappa} z_3 + \sqrt{r^2 - \left(\frac{\lambda}{\varkappa}\right)^2} - 2\sqrt{1 - \left(\frac{\lambda}{\varkappa}\right)^2}$$

$$z_1 = z_2 = |\theta_k| + \arcsin\frac{\lambda}{r} - 2\,\mathrm{acrsin}\,\lambda$$

$$z_3 = |\theta_k| + \arcsin\frac{\lambda}{r\varkappa} - 2\arcsin\frac{\lambda}{\varkappa}$$

$$\theta_k = \theta + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

The coefficients n_1 and n_2 must be made equal to zero, since the asymptotic expansions for the roots ω_1 and ω_2 are insufficiently accurate. The index k denotes the number of rotations performed by the corresponding emitted wave.

5.2. Diffracted waves. We write the inverse Fourier transformation of the LF-transformations (4.2) in the form

$$p_2^L(r, \theta, s) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} A_0(s) K_\omega(sr) \frac{I_\omega'(s) \exp(i\omega\theta)}{D_3 K_\omega'(s) - \delta_1 s D_2 K_\omega(s)} [1 + O(s^{-2/3})] d\omega \quad (5.7)$$

The integrand function in (5.7) has simple poles $\omega = \omega_k$ which are defined by the roots of the denominator. Substituting the Langer's asymptotic formulas [2, 28] into (5.7) and assuming that the parameter s is large and real, we obtain

$$\omega_k \sim \pm i [s + \alpha_k s^{1/3} + O(s^{-1/3})] \quad (5.8)$$

where α_k is obtained by the equation

$$\mathrm{Ai}'(x) = 0, \quad x = -2^{1/3}\alpha \quad (5.9)$$

Here $\mathrm{Ai}(x)$ is the Airy function, and the roots $x = a_k'$ of Eq. (5.9) can be found from tables [25].

It follows from (5.8) that the diffracted waves propagate at the velocity approximately equal to the speed of sound in the fluid. The elastic properties of the shell do not affect the coordinates of the poles ω_k to within the accuracy of the relation (5.8).

We use the Cauchy theorem on residues at the poles to evaluate the integral (5.7). Using the relations

$$\frac{\partial}{\partial\omega} K_\omega'(s) \sim -i\pi s^{-1} \frac{2^{2/3}}{3} f(\alpha) \exp\left(\frac{1}{2} i\pi\omega\right)$$

$$\left.\frac{\partial}{\partial\omega}\right|_{\omega=\omega_k} = -i s^{-1/3} \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\alpha_k}$$

the Langer's asymptotic formulas [2, 28], the formulas (5.8) and (5.9) we obtain, after simple manipulations,

$$p_2^L(r, \theta, s) \sim \sum_{k=1}^{n} i \; g^L(s) \gamma_1 \exp(\gamma_2 s - \gamma_3 s^{-1/3}) [s^{-1/6} - \delta_2 s^{-1/2} + O(s^{-2/3})] \quad (5.10)$$

$$\delta_2 = \frac{\delta}{8\alpha_k} \frac{1}{1 - \mu^2}, \quad \gamma = [2^{7/6} \pi^{1/2} (r^2 - 1)^{1/4} \alpha_k \mathrm{Ai}^2(-2^{-1/3}\alpha_k)]^{-1}$$

$$\gamma_2 = -|\theta| + \frac{\pi}{2} - 1 - (r^2 - 1)^{1/2} + \arccos\frac{1}{r}$$
$$\gamma_3 = \alpha_k\left(|\theta| - \frac{\pi}{2} - \arccos\frac{1}{r}\right)$$

We note that the values of the Airy functions $\mathrm{Ai}\,(x)$ corresponding to the roots of the equation $\mathrm{Ai}'\,(x) = 0$ can be found from tables [25].

6. Inverse Laplace transformation.

6.1. The reflected wave. Using the convolution and the displacement theorems, we find the original function in the L-transformation (5.5)

$$p_2(r,\theta,t) \sim p_*\left[g(\tau_*) - me^{-1/2 m\tau_*}\int_0^{\tau_*} e^{1/2 mx} g(x)\,dx + n\int_0^{\tau_*} g(x)\,dx\right] H(\tau_*),\quad \tau_* = t - a_* \quad (\tau_* \ll 1) \tag{6.1}$$

The second term in this expression explicitly represents the effect of the elastic properties of the shell on the reflected wave. The third term shows how the nondeformable convexity of the reflector affects the form of the reflected wave (as compared with the incident wave).

6.2. The emitted waves. Using the convolution and displacement theorems, we find the original function in the L-transformation (5.6)

$$p_2(r,\theta,t) = \sum_{j=1}^{3} P_j\left\{m_j\int_0^{\tau_j} g(x)\,v_{j0}(\tau_j - x)\,dx + n_j\int_0^{\tau_j} g(x)\,v_{j1}(\tau_j - x)\,dx\right\} H(\tau_j) \tag{6.2}$$

$$v_{10} = \left(\frac{z_1}{2\lambda}\right)^{-3/4} \tau_1^{3/4} J_{3/2}\left[2\left(\frac{z_1}{2\lambda}\tau_1\right)^{1/2}\right]$$
$$v_{20} = \left(\frac{\varkappa^2 z_2}{2a^2\lambda(1-\varkappa^2)}\right)^{-3/4} \tau_2^{3/4} J_{3/2}\left[2\left(\frac{\varkappa^2 z_2}{2a^2\lambda(1-\varkappa^2)}\tau_2\right)^{1/2}\right]$$
$$v_{11} = v_{21} = 0$$
$$v_{3\mu} = \frac{4^{\mu}\mu!}{(2\mu)!\sqrt{\pi}}\tau_3^{-1/2+\mu}\exp\left[-\frac{\delta_1 z_3}{2\lambda\sqrt{\varkappa^2 - \lambda^2}}\right],\quad \mu = 0, 1$$
$$\tau_j = t - l_j \quad (j = 1, 2, 3,\ \tau_j \ll 1)$$

The jump intensity across the front of emitted waves $\tau_1 = 0$ and $\tau_2 = 0$ is weaker than the jump intensity across the wavefront $\tau_3 = 0$ by two orders of magnitude. The elastic properties of the shell affect each of the peripheral emitted waves.

6.3. Diffracted waves. Using the convolution and displacment theorems we find the original function in the L-transformation (5.10)

$$p_2(r,\theta,t) \sim \sum_{k=1}^{n} p_0\gamma_1\left[\int_0^{\tau_k} g(x)\,u_{1/6}(\tau_0 - x)\,dx - \delta_2\int_0^{\tau_k} g(x)\,u_{1/2}(\tau_0 - x)\,dx\right] H(\tau_k) \tag{6.3}$$

$$u_l(\tau_0) = \frac{1}{2\pi i}\int_{a_0-i\infty}^{a_0+i\infty} s^{-l}\exp(\tau_0 s - \gamma_3 s^{1/3})\,ds,\ l = {}^1/_6, {}^1/_2$$

$$\tau_0 = t + \gamma_2 = t - 1 - |\theta| + \frac{\pi}{2} - (r^2-1)^{1/2} + \arccos\frac{1}{r}$$

$$\tau_k = \tau_0 + 2\pi(k-1), \quad \tau_k \ll 1$$

Following Friedlander, we evaluate the integrals $u(\tau_0)$, $(l = {}^1/_6, {}^1/_2)$ using the method of steepest descent. The coordinate of the point of steepest descent is given by the equation

$$\frac{\partial}{\partial s}(\tau_0 s - \gamma_3 s^{1/3}) = 0$$

and its value is $s_0 = (\gamma_3/3\tau_0)^{3/2}$. Substituting this value of s_0 into the standard formula of the steepest descent, we obtain the following approximate expressions:

$$u_l(\tau_0) = \frac{3^\alpha}{2\sqrt{\pi}}\gamma_3^{1-\alpha}\tau_0^{\alpha-1}\exp(-\gamma_4), \quad \alpha = \frac{6l-1}{4}, \quad \gamma_4 = \frac{2}{3^{3/2}}\frac{\gamma_3^{3/2}}{\tau_0^{1/2}} \qquad (6.4)$$

We note that the term containing δ_2 in (6.3) reflects explicitly the influence of the elastic properties of the shell on the diffracted waves. Figure 2 shows schematically the distribution of the wavefronts in the fluid at a fixed instant of time (since the pattern is symmetric, only the lower part is shown).

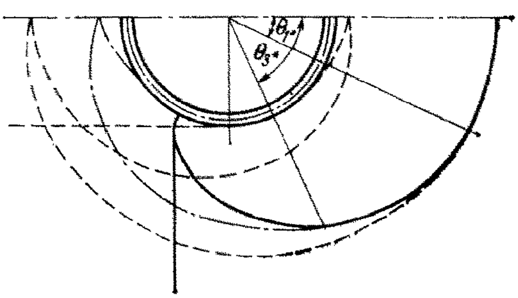

Fig. 2

Thus, using the method of repeated integral transformations we found the asymptotic solution of the problem near the pressure wavefronts. The independent variables r, θ and t can assume arbitrary values, but their combinations determining the distances from the wavefronts must invariably remain small.

For each type of wave we have obtained the following: an equation of the wavefront, the intensity of the pressure jump or its first derivatives at the wavefront, a formula describing the change in the pressure amplitude with increasing distance from the wavefront, and the variation in the pressure amplitude along the front. At small distances from the wavefronts the pressure amplitudes of all three wave types (reflected, emitted and diffracted) depend on the elastic properties of the shell. The strongest influence is exerted on the emitted wave generated by a shear wave propagating along the shell at the velocity C_2.

We use the asymptotic expansions obtained as a starting point to describe how the properties of the shell can be found from the system of waves "induced" by the shell. Let the source and sink of the acoustic waves be situated at a single point $(\theta = 0)$. As we know, the reflected wave can yield the following data: distance from the shell can be found from the lag of the reflected wave, and the shell radius from the purely geometrical reduction in the amplitude of the reflected wave compared with the transmitted wave. The formula (6.1) shows that we can also determine the relative wave resistance (drag) ζ_0 of the shell caused by the difference between the laws governing the pressure variation with time in the reflected and transmitted waves

$$\zeta_0 = \frac{\rho_1 C_I}{\rho C}, \quad C_I = \left[\frac{E(1-\nu)}{\rho_1(1+\nu)(1-2\nu)}\right]^{1/2}$$

If the sound source and receiver situated at the point ($\theta = 0$) are supplemented with a pressure detector at some other fixed point (with $\theta > \theta_{3*}$) , then from (6.2) it follows that we can determine the velocity of propagation C_2 of the shear waves in the shell from the delay of the emitted wave generated by the shear wave propagating through the shell at the velocity C_2.

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OPTIMIZATION PROBLEMS FOR PLATES OSCILLATING IN AN IDEAL FLUID

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The present paper deals with the oscillations of elastic plates in an ideal fluid. The optimizing problem of determining the thickness distribution for which the fundamental oscillation frequency is a maximum, is formulated. Necessary conditions for the extremum are derived. The relation between the fundamental frequency (a functional) and the parameters of the problem is investigated. The asymptotic behavior of the thickness and deflection distributions at the edges of the optimal plate is studied. An analytic solution of the optimization problem is given for thin, three-layer panels and it is shown that in this case the conditions of optimality are not only necessary, but also sufficient. The problem was